On stability of periodic solutions of the Navier-Stokes equations in unbounded domains(Structure of Solutions for Partial Differential Equations)

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On stability of periodic solutions of the Navier-Stokes equations in unbounded domains

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1 Introduction

Let Ω be an *exterior* domain in $\boldsymbol{R}^n (n \geq 3)$, the half space $\boldsymbol{R}^n_+$, or the whole space $\boldsymbol{R}^n$ and assume that the boundary $\partial\Omega$ is of class $C^{2+\mu}(0 < \mu < 1)$. The motion of the incompressible fluid occupying Ω is governed by the Navier-Stokes equations:

$$(N-S) \qquad \begin{cases} \frac{\partial w}{\partial t} - \Delta w + w \cdot \nabla w + \nabla \pi = f, \quad \mathrm{div}\, w = 0 \quad x \in \Omega, t \in \boldsymbol{R}, \\ w = 0 \quad \text{on } \partial\Omega, \qquad w(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

where $w = w(x,t) = (w^1(x,t), \cdots, w^n(x,t))$ and $\pi = \pi(x,t)$ denote the unknown velocity vector and the unknown pressure of the fluid, respectively, while $f = f(x,t) = (f^1(x,t), \cdots, f^n(x,t))$ is the given external force. In [11], Kozono-Nakao constructed periodic strong solutions in unbounded domains for some periodic external force f. Their solutions belong to $BC(\boldsymbol{R}; L^r \cap L^\infty)$ for some $n/2 < r < n$.

The purpose of the present paper is to show the *stability* of such solutions. If $w(x,0)$ is initially perturbed by a, then the perturbed flow $v(x,t)$ is governed by the following Navier-Stokes equations:

$$(N-S_1) \qquad \begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + v \cdot \nabla v + \nabla q = f, \quad \mathrm{div}\, v = 0 \quad \text{in } \Omega,\ t > 0, \\ v = 0 \quad \text{on } \partial\Omega, t > 0, \qquad v(x,t) \to 0 \quad \text{as } |x| \to \infty, \\ v(x,0) = w(x,0) + a(x) \quad \text{for } x \in \Omega. \end{cases}$$

We show that if the periodic solution w is small in $L^\infty(0,\infty; L^{m_1} \cap L^{m_2})$ for some $m_1 < n < m_2$ and if the initial disturbance a is small in $L^n(\Omega)$, then there is a unique *global strong solution* v of $(N-S_1)$ such that the integrals

$$\int_\Omega |v(x,t) - w(x,t)|^r dx \quad \text{for } n < r < \infty$$

converge to zero with *definite decay rates* as $t \to \infty$.

Let w and v be solutions of $(N-S_0)$ and $(N-S_1)$, respectively. Then the pair of functions $u \equiv v - w, p \equiv q - \pi$ satisfies

$$(N-S') \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + w \cdot \nabla u + u \cdot \nabla w + u \cdot \nabla u + \nabla p = 0 \quad \mathrm{div}\, u = 0 \quad \text{in} \Omega, t > 0, \\ u = 0 \quad \text{on } \partial\Omega, t > 0, \quad u(x,t) \to 0 \quad \text{as } |x| \to \infty, \quad u|_{t=0} = a. \end{cases}$$

Thus our problem on the stability for $(N-S)$ can now be reduced to investigation into existence of global strong solutions to $(N-S')$ and their asymptotic behavior. If $w \equiv 0$, our problem coincides with the initial boundary value problem for the usual nonstationary Navier-Stokes equations. Kato [9] constructed a global strong solution of $(N-S)$ having a decay property by the iteration method. His method needs the global estimate $\sup_{0<t<\infty} t^{1/2}\|\nabla u(t)\|_n < \infty$. On the other hand, the periodic solution w prevents us from getting this estimate. Hence we introduce a notion of *mild* solution as Kozono-Ogawa [13]. We first construct a global mild solution having a decay property. Then we shall show that this mild solution can be identified locally in time with the strong solution. Since the time interval of existence of strong solutions is characterized by the L^{2n}-norm of the initial data, we may conclude that our mild solution is actually a strong one.

2 Results

Throughout this paper we impose the following assumption on the domain.

Assumption 2.1 $\Omega \subset \boldsymbol{R}^n (n \geq 3)$ *is an exterior domain with smooth boundary, the half-space* $\boldsymbol{R}^n_+$ *or the whole space* $\boldsymbol{R}^n$.

Before stating our results, we introduce some notations and function spaces. Let $C^\infty_{0,\sigma}$ denote the set of all C^∞-real vector functions $\phi = (\phi^1, \cdots, \phi^n)$ with compact support in Ω such that div $\phi = 0$. L^r_σ is the closure of $C^\infty_{0,\sigma}$ with respect to the L^r-norm $\|\ \ \|_r$; $(\cdot,\cdot)$ denotes the L^2-inner product and the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. $\|\ \ \|_{r,\infty;T}$ and $\|\ \ \|_{r,\infty}$ denote the $L^\infty(0,T;L^r)$ and $L^\infty(0,\infty;L^r)$-norms, respectively. In this paper, we denote by C various constants. In particular, $C = C(*,\cdots,*)$ denotes the constant depending only on the quantities appearing in the parentheses.

Let us recall the Helmholtz decomposition:

$$L^r = L^r_\sigma \oplus G_r (\text{direct sum}), \quad 1 < r < \infty,$$

where $G_r = \{\nabla p \in L^r; p \in L^r_{loc}(\overline{\Omega})\}$. P_r denotes the projection operator from L^r onto L^r_σ along G_r. The Stokes operator A_r on L^r_σ is then defined by $A_r = -P_r\Delta$ with domain $D(A_r) = \{u \in W^{2,r}(\Omega); u|_{\partial\Omega} = 0\} \cap L^r_\sigma$.

Our definition of strong and mild solutions of (N-S) and (N-S') are as follows:

Definition 1 *Let* $a \in L^n_\sigma$. *A measurable function* u *on* $\Omega \times (0,T)$ *is called a strong solution of (N-S') on* $(0,T)$ *if*

(i) $u \in C([0,T);L^n_\sigma) \cap C^1((0,T);L^n_\sigma)$;

(ii) $u(t) \in D(A_n)$ *for* $t \in (0,T)$ *and* $A_n u \in C((0,T);L^n_\sigma)$;

(iii) u *satisfies*

$$\frac{\partial}{\partial t}u + A_n u + P_n(u\cdot\nabla u) + P(u\cdot\nabla w) + P(w\cdot\nabla u) = 0 \ \textit{in } L^n_\sigma \textit{ on } (0,T).$$

Similarly as above, for an external force $f \in C((0,T);L^n_\sigma)$ we define the strong solution of $(N-S)$ on $(0,T)$, so we do not write its definition here. Next we define a mild solution of (N-S') as Kozono-Ogawa [13]

Definition 2 *Let $a \in L^n_\sigma$ and let $w \in L^\infty(0,T; L^m_\sigma)$ for some $m > n$. Suppose that $n < r < \infty$. A measurable function u on $\Omega \times (0,T)$ is called a mild solution of $(N-S')$ in the class $S_r(0,T)$ if*

(i) $u \in BC([0,T); L^n_\sigma)$ *and* $t^{(1-n/r)/2}u(\cdot) \in BC([0,T); L^r_\sigma)$;

(ii) $\lim_{t\to+0} t^{(1-n/r)/2}\|u(t)\|_r = 0$;

(iii) u satisfies

$$\begin{aligned}(u(t),\phi) &= (e^{-tA}a,\phi) + \int_0^t (u(s)\cdot\nabla e^{-(t-s)A}\phi, u(s))ds \\ &\quad + \int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, u(s))ds \\ &\quad + \int_0^t (u(s)\cdot\nabla e^{-(t-s)A}\phi, w(s))ds\end{aligned}$$

for all $\phi \in C^\infty_{0,\sigma}$ and all $0 < t < T$.

Remark 2.1 By the similar argument given by Brezis [4] and Kato [10], we see that the condition (ii) follows from (i) and (iii), so (ii) is not necessary. The proof of this fact, however, is not brief. Hence we impose the condition (ii) for simplicity.

Our results are stated as follows.

Theorem 2.1 *Let $a \in L^n_\sigma$ and let $w(t) \in L^\infty(0,T; L^{m_1}_\sigma \cap L^{m_2}_\sigma)$ for some m_1, m_2 with $2n/(2n-3) \le m_1 < n < m_2$. There are positive numbers $\lambda_1(n,m_1,m_2), \lambda_2(n)$ such that if*

$$\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty} < 1/\lambda_1, \tag{2.1}$$

$$\|a\|_n < \lambda_2(1-\lambda_1(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty}))^2, \tag{2.2}$$

then there is a unique mild solution u of (N-S') in the class $S_{2n}(0,\infty)$ with the decay property

$$\|u(t)\|_l \le Ct^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{l})} \text{ for } n \le l \le 2n.$$

Theorem 2.2 *Let (2.1) and (2.2) hold. For every $2n < r < \infty$, there are positive numbers $\eta_1(n,m_1,m_2,r), \eta_2(n,r)$ such that if*

$$\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty} < 1/\eta_1, \tag{2.3}$$

$$\|a\|_n < \eta_2(1-\eta_1(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty}))^2, \tag{2.4}$$

then the mild solution u given in Theorem 2.1 has the additional decay property

$$\|u(t)\|_l \le Ct^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{l})} \text{ for } 2n \le l \le r.$$

Theorem 2.3 *In addition to the hypotheses of Theorem 2.1, assume moreover that w is a strong solution of (N-S) on $(0,\infty)$ for some external force $f \in C((0,\infty); L^n_\sigma)$. Then the mild solution given in Theorem 2.1 is a strong solution of (N-S') on $(0,\infty)$.*

Remark 2.2. When $\Omega = \boldsymbol{R}^n$ with $n \ge 3$ and when Ω is an exterior domain in $\boldsymbol{R}^n$ with $n \ge 4$, for small periodic force f, Kozono-Nakao [11] constructed the strong periodic solution w with (2.1); their solution w belongs to $BC(\boldsymbol{R}; L^r)$ for $2 < r < n$ with $\nabla w \in BC(\boldsymbol{R}; L^q)$ for $n/2 < q < n$. If f is sufficiently small, then $\|w\|_{L^\infty(0,\infty;L^r)} + \|\nabla w\|_{L^\infty(0,\infty;L^q)}$ is also sufficiently small. By the Sobolev inequality, $w \in BC(\boldsymbol{R}; L^p)$ for all $p \in [r, nq/(n-q)]$. Since $nq/(n-q) > n$, this implies (2.1).

Maremonti [14],[15] also showed the existence of the periodic solutions in the three-dimensinal whole space $\boldsymbol{R}^3$ and the half space $\boldsymbol{R}^3_+$. It seems to be an open question wether there exists a periodic solution in *three-dimensional exterior* domain.

3 Preliminaries.

Let us first recall the following $L^q - L^r$-estimate for the semigroup $\{e^{-tA}\}_{t\geq 0}$.

Lemma 3.1 (Kato[9], Ukai[17], Giga-Sohr[7], Iwasita[8], Borchers-Miyakawa[1],[2])

$$\|e^{-tA}a\|_r \leq M_{q,r}t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|a\|_q, \quad 1 < q \leq r < \infty, \tag{3.1}$$

$$\|\nabla e^{-tA}a\|_r \leq M'_{q,r}t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|a\|_q, \quad 1 < q \leq r \leq n \tag{3.2}$$

for all $a \in L^q_\sigma$ and all $t > 0$, where $M_{q,r}, M'_{q,r}$ are constants depending only on q, r.

Concerning $r = \infty$, we have

Lemma 3.2 (Chen[5],Borchers-Miyakawa[1],[3])

$$\|e^{-tA}a\|_\infty \leq M_{q,\infty}t^{-\frac{n}{2q}}\|a\|_q, \quad 1 < q \leq 2n, \tag{3.3}$$

for all $a \in L^q_\sigma$ and all $t > 0$, with the constant $M_{q,\infty}$ depending only on q.

By Lemma 3.1, we have the following lemmas.

Lemma 3.3 *Let $0 < T \leq \infty$. Suppose that u is a measurable function with $t^{\frac{1-\alpha}{2}}u(\cdot) \in L^\infty(0,T;L^{n/\alpha}_\sigma)$ for some $0 < \alpha < 1$ and that $w \in L^\infty(0,T;L^{m_1}_\sigma \cap L^{m_2}_\sigma)$ for some m_1, m_2 with $\frac{n}{n-\alpha-1} \leq m_1 < n < m_2$. Then there holds*

$$\begin{aligned}
&\left|\int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, u(s))ds\right| + \left|\int_0^t (u(s)\cdot\nabla e^{-(t-s)A}\phi, w(s))ds\right| \\
\leq\ & C(\alpha, m_1, m_2, n)(\|w\|_{m_1,\infty;T} + \|w\|_{m_2,\infty;T})(\sup_{0<s<t} s^{\frac{1-\alpha}{2}}\|u(s)\|_{n/\alpha})t^{\frac{\alpha-1}{2}}\|\phi\|_{\frac{n}{n-\alpha}}
\end{aligned}$$

for all $0 < t < T$.

Lemma 3.4 *Let $0 < T \leq \infty$ and let v and w be measurable functions with $w \in L^\infty(0,T;L^{n/\gamma}_\sigma)$ and $t^{\frac{1-\alpha}{2}}v(\cdot) \in L^\infty(0,T;L^{n/\alpha}_\sigma)$ for some $0 < \gamma, \alpha < 1$. Then for $\delta \in [\alpha, \alpha+\gamma]$ and $0 < \beta < \frac{1}{2} + \frac{\delta}{2} - \frac{\alpha}{2} - \frac{\gamma}{2} (> 0)$,*

$$\begin{aligned}
F_{w,v}(t,h) \ &\equiv\ \left|\int_0^{t+h}(w(s)\cdot\nabla e^{-(t+h-s)A}\phi, v(s))ds - \int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, v(s))ds\right| \\
&\leq\ C\left(\sup_{0<s<T}\|w(s)\|_{n/\gamma}\right)\left(\sup_{0<s<T} s^{\frac{1-\alpha}{2}}\|v(s)\|_{n/\alpha}\right) \\
&\quad\times(h^\beta t^{\frac{\delta}{2}-\frac{\gamma}{2}-\alpha-\beta} + h^{\frac{1}{2}+\frac{\delta}{2}-\frac{\alpha}{2}-\frac{\gamma}{2}}t^{\frac{-1+\alpha}{2}})\|\phi\|_{\frac{n}{n-\delta}} \\
F_{v,w}(t,h) \ &\equiv\ \left|\int_0^{t+h}(v(s)\cdot\nabla e^{-(t+h-s)A}\phi, w(s))ds - \int_0^t (v(s)\cdot\nabla e^{-(t-s)A}\phi, w(s))ds\right| \\
&\leq\ C\left(\sup_{0<s<T}\|w(s)\|_{n/\gamma}\right)\left(\sup_{0<s<T} s^{\frac{1-\alpha}{2}}\|v(s)\|_{n/\alpha}\right) \\
&\quad\times(h^\beta t^{\frac{\delta}{2}-\frac{\gamma}{2}-\alpha-\beta} + h^{\frac{1}{2}+\frac{\delta}{2}-\frac{\alpha}{2}-\frac{\gamma}{2}}t^{\frac{-1+\alpha}{2}})\|\phi\|_{\frac{n}{n-\delta}},
\end{aligned}$$

for all $h > 0$ *and* $0 < t < t+h < T$, *where* C *is independent of* w, v, ϕ *and* T. *For* $\delta \in [\alpha, 2\alpha]$ *and* $0 < \beta < \frac{1}{2} - \alpha + \frac{\delta}{2} (> 0)$,

$$\begin{aligned} F_{v,v}(t,h) &\equiv \left| \int_0^{t+h} (v(s) \cdot \nabla e^{-(t+h-s)A}\phi, v(s))ds - \int_0^t (v(s) \cdot \nabla e^{-(t-s)A}\phi, v(s))ds \right| \\ &\leq C \left(\sup_{0<s<T} s^{\frac{1-\alpha}{2}} \|v(s)\|_{n/\alpha} \right)^2 (h^\beta t^{\frac{\delta}{2}-\frac{1}{2}-\beta} + h^{\frac{1}{2}-\alpha+\frac{\delta}{2}} t^{-1+\alpha}) \|\phi\|_{\frac{n}{n-\delta}}, \end{aligned}$$

for all $h > 0$ *and* $0 < t < t+h < T$.

Concerning the mild solution, we have

Lemma 3.5 *Let* $h \in (0,T)$ *and let* u *be a mild solution of* $(N-S')$ *in the class* $S_r(0,T)$, $(n < r < \infty)$. *Then* $u(\cdot + h)$ *is also a mild solution of* $(N-S')$ *in the class* $S_r(0, T-h)$ *with initial data* $u(h)$.

Concerning the uniqueness of mild solutions, we have

Lemma 3.6 (Uniqueness) *Let* $a \in L^n_\sigma$ *and let* $w \in L^\infty(0,T; L^m_\sigma)$ *for some* $m > n$. *Suppose that* $n < r < \infty$. *Then the mild solution of* $(N-S')$ *is unique within the class* $S_r(0,T)$.

Proof. Following [13] we give the proof. Let u and v be mild solutions of $(N-S')$ in $S_r(0,T)$ with tha same initial data a. Set

$$\begin{aligned} D(t) &\equiv \sup_{0<s\leq t} \|u(s) - v(s)\|_n \\ K(t) &\equiv \sup_{0<s\leq t} s^{(1-\beta)/2} \|u(s)\|_{n/\beta} + \sup_{0<s\leq t} s^{(1-\beta)/2} \|v(s)\|_{n/\beta}, \end{aligned}$$

where $\beta = n/r$. Similarly to the proof of Lemma 3.3, we have by (iii) in Definition 2 and Lemma 3.1 that

$$|(u(t) - v(t), \phi)| \leq \left\{ C_* K(t) + B_* t^{\frac{1}{2}(1-\frac{n}{m})} \right\} D(t) \|\phi\|_{\frac{n}{n-1}},$$

for all $\phi \in C^\infty_{0,\sigma}$ and all $0 < t < T$, where $C_* = M'_{\frac{n}{n-1}, \frac{n}{n-1-\beta}} B(\frac{1-\beta}{2}, \frac{1+\beta}{2})$ and $B_* = \frac{4m}{m-n} M'_{\frac{n}{n-1}, \delta} \|w\|_{m,\infty;T}$, $(1/\delta = 1 - 1/m - 1/n)$. By duality we have

$$D(t) \leq (C_* K(t) + B_* t^{\frac{1}{2}(1-\frac{n}{m})}) D(t), \quad 0 < t < T.$$

Since $\lim_{t\to+0} K(t) = 0$, we can choose small positive number t_0 such that $D(t_0) \leq \frac{1}{2} D(t_0)$, which implies

$$u(t) \equiv v(t) \quad \text{for } 0 \leq t \leq t_0.$$

Next we show that $u(t) \equiv v(t)$ for $t_0 \leq t < T$, by Lemma 3.5. Let

$$\begin{aligned} D^h(t) &\equiv \sup_{0\leq s\leq t} \|u(s+h) - v(s+h)\|_n, \\ K^h(t) &\equiv \sup_{0\leq s\leq t} s^{(1-\beta)/2} \|u(s+h)\|_{n/\beta} + \sup_{0\leq s\leq t} s^{(1-\beta)/2} \|v(s+h)\|_{n/\beta}, \\ K_* &\equiv \sup_{0\leq s\leq T} s^{(1-\beta)/2} \|u(s)\|_{n/\beta} + \sup_{0\leq s\leq T} s^{(1-\beta)/2} \|v(s)\|_{n/\beta}, \end{aligned}$$

for $0 < t < t+h < T$. We easily show

$$K^h(t) \le K_* h^{\frac{-1+\beta}{2}} t^{\frac{1-\beta}{2}} \le K_* t_0^{\frac{-1+\beta}{2}} t^{\frac{1-\beta}{2}}$$

for all $h \ge t_0$ and all $0 < t < T-h$.

Suppose that $u(t_1) \equiv v(t_1)$ for some $t_1 \ge t_0$. Then, by lemma 3.5 we see that $u(\cdot + t_1)$ and $v(\cdot + t_1)$ is mild solutions in the class $S_r(0, T-t_1)$ with same initial data $u(t_0)$. By the above argument we have

$$D^{t_1}(t) \le (C_* K^{t_1}(t) + B_* t^{\frac{1}{2}(1-\frac{n}{m})}) D^{t_1}(t), \quad 0 < t < T - t_1.$$

Letting $\xi \equiv \min\{1/(4C_* t_0^{\frac{-1+\beta}{2}} K_*)^{\frac{2}{\beta-1}}, 1/(4B_*)^{\frac{2m}{m-n}}\}$, we obtain $D^{t_1}(\xi) \le \frac{1}{2} D^{t_1}(\xi)$ which implies

$$u(t) \equiv v(t) \quad \text{for } t_1 \le t \le t_1 + \xi.$$

Since ξ can be choosen independent of t_1, we can repeat the same argument as above for $t \ge t_1 + \xi$ and we have $u(t) \equiv v(t)$ for all $t \in [0, T)$. This proves Lemma 3.6.

4 Proof of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Let us construct the mild solution according to the following scheme:

$$(4.1) \qquad u_0(t) = e^{-tA} a,$$

$$(4.2) \qquad \begin{aligned} (u_{j+1}(t), \phi) = {} & (e^{-tA} a, \phi) + \int_0^t (u_j(s) \cdot \nabla e^{-(t-s)A} \phi, u_j(s)) ds \\ & + \int_0^t (w(s) \cdot \nabla e^{-(t-s)A} \phi, u_j(s)) ds \\ & + \int_0^t (u_j(s) \cdot \nabla e^{-(t-s)A} \phi, w(s)) ds, \quad j = 0, 1 \ldots. \end{aligned}$$

for all $\phi \in C_{0,\sigma}^\infty$ and all $0 < t < \infty$. Indeed, we can see that there is a function u_{j+1} satisfying (4.2) with $t^{1/4} u_{j+1}(\cdot) \in L^\infty(0,\infty); L_\sigma^{2n})$ if $t^{1/4} u_j(\cdot) \in L^\infty(0,\infty); L_\sigma^{2n})$. To see this, we assume that

$$(4.3) \qquad \sup_{0<t<\infty} t^{\frac{1-\alpha}{2}} \|u_j(t)\|_{\frac{n}{\alpha}} \le K_{\alpha,j} < \infty \quad \text{for some } 0 < \alpha \le 1/2.$$

From Lemma 3.1, we obtain

$$(4.4) \qquad \left| \int_0^t (u_j(s) \cdot \nabla e^{-(t-s)A} \phi, u_j(s)) ds \right| \le M'_{\frac{n}{n-\alpha}, \frac{n}{n-2\alpha}} (K_{\alpha,j})^2 B(\alpha, \tfrac{1-\alpha}{2}) t^{\frac{\alpha-1}{2}} \|\phi\|_{\frac{n}{n-\alpha}}$$

for all $\phi \in C_{0,\sigma}^\infty$ and all $0 < t < \infty$. By Lemma 3.3, we have

$$(4.5) \qquad \begin{aligned} & \left| \int_0^t (w(s) \cdot \nabla e^{-(t-s)A} \phi, u_j(s)) ds \right| + \left| \int_0^t (u_j(s) \cdot \nabla e^{-(t-s)A} \phi, w(s)) ds \right| \\ \le {} & C(\alpha, m_1, m_2, n)(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty}) (\sup_{0<s<t} s^{\frac{1-\alpha}{2}} \|u_j(s)\|_{n/\alpha}) t^{\frac{\alpha-1}{2}} \|\phi\|_{\frac{n}{n-\alpha}} \end{aligned}$$

for all $0 < t < \infty$.

Obviously we have

$$(4.6) \qquad |(e^{-tA} a, \phi)| \le \|e^{-tA} a\|_{\frac{n}{\alpha}} \|\phi\|_{\frac{n}{n-\alpha}} \le M_{n,\frac{n}{\alpha}} t^{\frac{\alpha-1}{2}} \|a\|_{\frac{n}{\alpha}} \|\phi\|_{\frac{n}{n-\alpha}}$$

Hence it follows from (4.4),(4.5),(4.6) and duality that under the assumption (4.3), there is a unique function $u_{j+1}(t) \in L_\sigma^{n/\alpha}$ satisfying (4.2) for all $t > 0$ with

$$(4.7) \qquad \sup_{0<t<\infty} t^{\frac{\alpha-1}{2}} \|u_{j+1}(t)\|_{\frac{n}{\alpha}} \leq M_{n,\frac{n}{\alpha}}\|a\|_n + C_1(\alpha,n)(K_{\alpha,j})^2 + C_2(\alpha,m_1,m_2,n)(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty})K_{\alpha,j}.$$

Now we have

$$\sup_{0<t<\infty} t^{\frac{1-\alpha}{2}} \|u_0(t)\|_{\frac{n}{\alpha}} = \sup_{0<t<\infty} t^{\frac{1-\alpha}{2}} \|e^{-tA}a\|_{\frac{n}{\alpha}} \leq M_{n,\frac{n}{\alpha}}\|a\|_n,$$

which show (4.3) is true for $j = 0$ with $K_{\alpha,0} = M_{n,\frac{n}{\alpha}}\|a\|_n$. Therefore by induction we see that for all $j = 0, 1 \dots$, there is a unique function u_{j+1} satisfying (4.2) and (4.3) with j replaced by $j+1$ and that

$$(4.8) \qquad K_{\alpha,j+1} = K_{\alpha,0} + C_1(K_{\alpha,j})^2 + C_2(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty})K_{\alpha,j}$$

Moreover, we can see that $u_j \in C(0,\infty; L_\sigma^{n/\alpha})$. Indeed we have

$$(u(t+h) - u(t), \phi) = ((e^{-hA} - 1)e^{-tA}a, \phi) + F_{u_j,u_j}(t,h) + F_{w,u_j}(t,h) + F_{u_j,w}(t,h)$$

for all $\phi \in C_{0,\sigma}^\infty$ and all $0 < t < t+h$, where $F_{u,v}(t,h)$ is defined in Lemma 3.4. From Lemma 3.1 we obtain

$$|((e^{-hA} - 1)e^{-tA}a, \phi)| \leq C(\alpha,\beta,n)h^\beta t^{-\beta-\frac{1}{2}+\frac{\alpha}{2}}\|a\|_n\|\phi\|_{\frac{n}{n-\alpha}}, (0 < \beta < 1)$$

Hence from this estimate, Lemma 3.4 and duality it follows that $u_j \in C(0,\infty : L_\sigma^{n/\alpha})$.
If we assume for some $0 < \alpha \leq 1/2$ that

$$(4.9) \qquad C_2(\alpha,m_1,m_2,n)(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty}) < 1;$$

$$(4.10) \qquad 4M_{n,\frac{n}{\alpha}}\|a\|_n C_1(\alpha,n) < (1 - C_2(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty}))^2,$$

then the sequence $\{K_{\alpha,j}\}_{j=o}^\infty$ is bounded with

$$(4.11) \qquad K_{\alpha,j} < \frac{1 - C_2|||w||| - \sqrt{(1 - C_2|||w|||)^2 - 4K_{\alpha,0}C_1(\alpha,n)}}{2C_1(\alpha,n)} \equiv k_\alpha, \quad j = 01, \dots,$$

where $|||w||| \equiv \|w\|_{m_1,\infty} + \|w\|_{m_2,\infty}$. From now on we assume (4.9) and (4.10) for some $0 < \alpha \leq 1/2$. Set $v_j \equiv u_j - u_{j-1}(u_{-1} \equiv 0)$. By Lemma 3.3 we see that

$$(4.12) \qquad |(v_{j+1}(t), \phi)| \leq (2C_1 k_\alpha + C_2|||w|||)\left(\sup_{0<s<\infty} s^{\frac{1-\alpha}{2}}\|v_j(s)\|_{\frac{n}{\alpha}}\right) t^{\frac{\alpha-1}{2}}\|\phi\|_{\frac{n}{n-\alpha}}.$$

Letting $C_{\alpha,3} \equiv 2C_1(\alpha,n)k_\alpha + C_2(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty})$, from duality we obtain

$$\sup_{0<s<\infty} s^{\frac{1-\alpha}{2}}\|v_{j+1}(s)\|_{\frac{n}{\alpha}} \leq C_{\alpha,3}\left(\sup_{0<s<\infty} s^{\frac{1-\alpha}{2}}\|v_j(s)\|_{\frac{n}{\alpha}}\right), \quad j = 0, 1, \dots,$$

which yields

$$(4.13) \qquad \sup_{0<s<\infty} s^{\frac{1-\alpha}{2}}\|v_j(s)\|_{\frac{n}{\alpha}} \leq (C_{\alpha,3})^j\left(\sup_{0<s<\infty} s^{\frac{1-\alpha}{2}}\|v_0(s)\|_{\frac{n}{\alpha}}\right) \leq M_{n,\frac{n}{\alpha}}\|a\|_n(C_{\alpha,3})^j.$$

Since (4.11) implies $0 < C_{\alpha,3} < 1$ and since $u_j = \sum_{i=0}^{j} v_i$, (4.13) yields a limit $u \in C((0,\infty); L_\sigma^{n/\alpha})$ with $t^{\frac{1-\alpha}{2}} u(\cdot) \in BC((0,\infty); L_\sigma^{n/\alpha})$ such that

$$(4.14) \qquad \sup_{0<t<\infty} t^{\frac{1-\alpha}{2}} \|u_j(t) - u(t)\|_{\frac{n}{\alpha}} \to 0 \text{ as } j \to \infty.$$

Following Kozono-Ogawa [13], we can show $\lim_{t\to+0} t^{\frac{1-\alpha}{2}} \|u(t)\|_{\frac{n}{\alpha}} = 0$. Indeed it follows that

$$(4.15) \qquad \begin{aligned} \sup_{0<t<T} t^{\frac{1-\alpha}{2}} \|e^{-tA}a\|_{\frac{n}{\alpha}} &\le \sup_{0<t<T} t^{\frac{1-\alpha}{2}} \|e^{-tA}(a-\tilde{a})\|_{\frac{n}{\alpha}} + \sup_{0<t<T} t^{\frac{1-\alpha}{2}} \|e^{-tA}\tilde{a}\|_{\frac{n}{\alpha}} \\ &\le M_{n,\frac{n}{\alpha}} \|a-\tilde{a}\|_n + M_{\frac{n}{\alpha},\frac{n}{\alpha}} \|\tilde{a}\|_{\frac{n}{\alpha}} T^{\frac{1-\alpha}{2}} \end{aligned}$$

for all $\tilde{a} \in L_\sigma^n \cap L_\sigma^{2n}$ and all $0 < T < \infty$. Since (4.3)-(4.11) hold with $0 < t < \infty$ replaced by $0 < t < T$ for arbitrary $T > 0$ and since $L_\sigma^n \cap L_\sigma^{2n}$ is dense in L_σ^n, (4.11) with the aid of (4.15) yields

$$(4.16) \qquad \sup_{0<t<T} t^{\frac{1-\alpha}{2}} \|u_j(t)\|_{\frac{n}{\alpha}}, \ \sup_{0<t<T} t^{\frac{1-\alpha}{2}} \|u(t)\|_{\frac{n}{\alpha}} \to 0 \text{ as } T \to 0$$

We next show $u \in BC([0,\infty)); L_\sigma^n)$ if (4.9) and (4.10) hold for $\alpha = 1/2$. From now on we assume that (4.9) and (4.10) hold for $\alpha = 1/2$. Since $w \in L^\infty(0,\infty; L_\sigma^{m_1} \cap L_\sigma^{m_2})$, we can take $0 < \gamma < 1$ such that $\alpha + \gamma \ge 1$ and $w \in L^\infty(0,\infty; L_\sigma^{n/\gamma})$. Then, in the similar way to proving $u_j \in C((0,\infty); L_\sigma^{n/\alpha})$, by Lemma 3.4 (with $\delta = 1$) and duality, we have $u_j \in C((0,\infty); L_\sigma^n)$. From Lemma 3.1, we obtain

$$\begin{aligned} \|u_0(t)\|_n &\le M_{n,n}\|a\|_n \\ \left|\int_0^t (u_j(s)\cdot\nabla e^{-(t-s)A}\phi, u_j(s))ds\right| &\le M'_{\frac{n}{n-1},\frac{n}{n-1}} (k_{\frac{1}{2}})^2 B(\tfrac{1}{2},\tfrac{1}{2}) \|\phi\|_{\frac{n}{n-1}}, \\ \left|\int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, u_j(s))ds\right| &\le M'_{\frac{n}{n-1},\frac{2n}{2n-3}} \|w\|_{n,\infty} (k_{\frac{1}{2}}) B(\tfrac{1}{4},\tfrac{3}{4}) \|\phi\|_{\frac{n}{n-1}}, \\ \left|\int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, u_j(s))ds\right| &\le M'_{\frac{n}{n-1},\frac{2n}{2n-3}} \|w\|_{n,\infty} (k_{\frac{1}{2}}) B(\tfrac{1}{4},\tfrac{3}{4}) \|\phi\|_{\frac{n}{n-1}}, \end{aligned}$$

for all $\phi \in C_{0,\sigma}^\infty$, $t > 0$, which yield the following uniform estimate:

$$\sup_{0<t<\infty} \|u_{j+1}\|_n \le M_{n,n}\|a\|_n + M'_{\frac{n}{n-1},\frac{n}{n-1}} (k_{\frac{1}{2}})^2 B(\tfrac{1}{2},\tfrac{1}{2}) + 2M'_{\frac{n}{n-1},\frac{2n}{2n-3}} \|w\|_{n,\infty} k_{\frac{1}{2}} B(\tfrac{1}{4},\tfrac{3}{4}).$$

Concerning continuity of u_j at $t = 0$ in L_σ^n, as above we obtain

$$\begin{aligned} \|u_{j+1}(t) - a\|_n \le\ & \|e^{-tA}a - a\|_n + M'_{\frac{n}{n-1},\frac{n}{n-1}} \left(\sup_{0<s<t} s^{1/4}\|u_j(s)\|_{2n}\right)^2 B(\tfrac{1}{2},\tfrac{1}{2}) \\ & + 2M'_{\frac{n}{n-1},\frac{2n}{2n-3}} \|w\|_{n,\infty} \left(\sup_{0<s<t} s^{1/4}\|u_j(s)\|_{2n}\right) B(\tfrac{1}{4},\tfrac{3}{4}), \end{aligned}$$

which yields with the aid of (4.16) $\lim_{t\to+0} \|u_j(t) - a\|_n = 0$. Concerning $v_j (\equiv u_j - u_{j-1})$, as (4.12) we have

$$\begin{aligned} |(v_{j+1}(t),\phi)| \le\ & 2M'_{\frac{n}{n-1},\frac{n}{n-1}} k_{1/2} B(\tfrac{1}{2},\tfrac{1}{2}) \left(\sup_{0<s<\infty} s^{\frac{1}{4}}\|v_j(s)\|_{2n}\right) \|\phi\|_{\frac{n}{n-1}} \\ & + 2M'_{\frac{n}{n-1},\frac{2n}{2n-1}} \|w\|_{n,\infty} B(\tfrac{3}{4},\tfrac{1}{4}) \left(\sup_{0<s<\infty} s^{\frac{1}{4}}\|v_j(s)\|_{2n}\right) \|\phi\|_{\frac{n}{n-1}}, \end{aligned}$$

which implies by duality that

$$\sup_{0<s<\infty} \|v_{j+1}(s)\|_n \le C(n,w,k_{1/2}) \sup_{0<s<\infty} s^{\frac{1}{4}}\|v_j(s)\|_{2n} \text{ for } j=0,1,.... \tag{4.17}$$

From this and (4.13) with $\alpha = 1/2$ we obtain

$$\begin{aligned} \sup_{0<s<\infty} \|u_l(s)-u_m(s)\|_n &= \sup_{0<s<\infty} \|\sum_{j=m+1}^{l} v_j(s)\|_n \\ &\le CM_{n,2n}\|a\|_n \sum_{j=m}^{l-1} (C_{\alpha,3})^j \text{ for } , l>m\ge 0. \end{aligned} \tag{4.18}$$

Hence it follows from (4.18) and $0 < C_{\alpha,3} < 1$ that the limit u belongs to $u \in BC([0,\infty); L^n_\sigma)$. To see that u is desired mild solution of (N-S') in the class $S_{2n}(0,\infty)$, we need to prove that u satisfies (iii) in Definition 2. By Lemma 3.1 and (4.14), we have

$$\begin{aligned} & \left|\int_0^t (u_j(s)\cdot\nabla e^{-(t-s)A}\phi, u_j(s))ds - \int_0^t (u(s)\cdot\nabla e^{-(t-s)A}\phi, u(s))ds\right| \\ \le\ & \int_0^t (\|u_j(s)\|_{2n}+\|u(s)\|_{2n})\|u_j(s)-u(s)\|_{2n}\|\nabla e^{-(t-s)A}\phi\|_{\frac{n}{n-1}}ds \\ \le\ & 2M'_{\frac{n}{n-1},\frac{n}{n-1}}k_{\frac{1}{2}} \sup_{0<s<\infty} s^{\frac{1}{4}}\|u_j(s)-u(s)\|_{2n}B(\tfrac{1}{2},\tfrac{1}{2})\|\phi\|_{\frac{n}{n-1}} \\ & \to 0 \text{ as } j\to\infty \quad (\phi\in C^\infty_{0,\sigma}), \\ & \left|\int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, u_j(s))ds - \int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, u(s))ds\right| \\ \le\ & M'_{\frac{n}{n-1},\frac{2n}{2n-3}}\|w\|_{n,\infty} \sup_{0<s<\infty} s^{\frac{1}{4}}\|u_j(s)-u(s)\|_{2n}B(\tfrac{1}{4},\tfrac{3}{4})\|\phi\|_{\frac{n}{n-1}} \\ & \to 0 \text{ as } j\to\infty \quad (\phi\in C^\infty_{0,\sigma}), \\ & \left|\int_0^t (u_j(s)\cdot\nabla e^{-(t-s)A}\phi, w(s))ds - \int_0^t (u(s)\cdot\nabla e^{-(t-s)A}\phi, w(s))ds\right| \\ & \to 0 \text{ as } j\to\infty \quad (\phi\in C^\infty_{0,\sigma}), \end{aligned}$$

which yield (iii) in Definition 2. Now it remains to show that

$$\|u(t)\|_l \le Ct^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{l})} \quad \text{for } n\le l\le 2n.$$

Since $u \in L^\infty(0,\infty;L^n)$ and $t^{1/4}u(\cdot) \in L^\infty(0,\infty;L^{2n})$, we get this estimate by the Hölder inequality. This completes the proof of Theorem 2.1.

As for the proof of Theorem 2.2, we have $t^{\frac{1-n/r}{2}}u(\cdot) \in L^\infty(0,\infty;L^r_\sigma)$, provided (4.9) and (4.10) hold for $\alpha = n/r$. The remaining argument is similar to the above. This proves Theorem 2.2.

5 Proof of Theorem 2.3.

Let $L^\infty_{loc}([0,\infty);L^n)$ denote the set of all measurable functions u such that $u \in L^\infty(0,T;L^n)$ for all $T > 0$. To prove Theorem 2.3, We need the following local existence theorem:

Theorem 5.1 (Local existence) *Let $a \in L^n_\sigma \cap L^{n/\alpha}_\sigma$ for some $\alpha \in (0,1)$ and let w be a measurable function on $(0,\infty)$ with $w \in L^\infty(0,\infty; L^m_\sigma)$ for some $m > n$ and $t^{1/2}\nabla w(\cdot) \in L^\infty_{loc}([0,\infty); L^n)$. Then there exists a mild solution u of $(N-S')$ in the class $S_{n/\alpha}(0,T^*)$ satisfying*

$$u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A}P(u\cdot\nabla u + w\cdot\nabla u + u\cdot\nabla w)(s)ds \text{ in } L^n_\sigma$$

where

$$\begin{aligned}
T^* &= \min\left\{\left[\frac{1}{16(C_1+C_4)M_{\frac{n}{\alpha},\frac{n}{\alpha}}\|a\|_{\frac{n}{\alpha}}}\right]^{\frac{2}{1-\alpha}}, \left(\frac{1}{2(C_4+C_5)\|w\|_{m,\infty}}\right)^{\frac{2m}{m-n}}\right\},\\
C_1 &= C_1(\alpha,n) = M'_{\frac{n}{n-\alpha},\frac{n}{n-2\alpha}}B(\alpha,\tfrac{1-\alpha}{2})\\
C_4 &= Q_{\frac{n}{\alpha+1}}M'_{\frac{n}{\alpha+1},n}B(\tfrac{1-\alpha}{2},\tfrac{\alpha}{2}) + Q_{\frac{nm}{n+m}}M'_{\frac{nm}{n+m},n}B(\tfrac{1}{2}(1-\tfrac{n}{m}),\tfrac{1}{2}), \quad Q_l = \|P_l\|_{B(L^l,L^l_\sigma)},\\
C_5 &= 2M'_{\frac{n}{n-\alpha},\frac{mn}{mn-m\alpha-n}}B(\tfrac{\alpha+1}{2},\tfrac{1}{2}(1-\tfrac{n}{m})).
\end{aligned}$$

Moreover if there is positive number $\kappa \in (0,1)$ such that

$$w \in C^\kappa([\xi,T^*]; L^\infty), \quad \nabla w \in C^\kappa([\xi,T^*]; L^n)$$

for all $\xi \in (0,T^*)$, then u is also a strong solution of $(N-S')$ on $(0,T^*)$.

Remark. In case $w \equiv 0$, the existence interval T^* was obtained by Giga [6].

Proof of Theorem 5.1. Let us construct the strong solution according to the following scheme:

$$u_0(t) = e^{-tA}a, \tag{5.1}$$

$$\begin{aligned}
u_{j+1}(t) = \; & e^{-tA}a - \int_0^t e^{-(t-s)A}P(u_j\cdot\nabla u_j)(s)ds\\
& - \int_0^t e^{-(t-s)A}P(w\cdot\nabla u_j)(s)ds - \int_0^t e^{-(t-s)A}P(u_j\cdot\nabla w)(s)ds.
\end{aligned} \tag{5.2}$$

Then we can see that for $0 < T < \infty$

$$\sup_{0<t<T} t^{\frac{1-\alpha}{2}}\|u_j(t)\|_{n/\alpha} \le K^T_{\alpha,j} < \infty, \quad j = 0,1,..., \tag{5.3}$$

$$\sup_{0<t<T} t^{\frac{1}{2}}\|\nabla u_j(t)\|_n \le L^T_j < \infty, \quad j = 0,1,.... \tag{5.4}$$

Suppose that (5.3) and (5.4) are true. Then, multiplying (5.2) by ϕ and integrating by parts, we obtain the identity (4.2). We have by (3.2) and the Hölder inequality that

$$\begin{aligned}
&\left|\int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, u_j(s))ds\right| + \left|\int_0^t (u_j(s)\cdot\nabla e^{-(t-s)A}\phi, w(s))ds\right|\\
\le \; & C_5\|w\|_{m,\infty;T}(\sup_{0<s<t} s^{\frac{1-\alpha}{2}}\|u_j(s)\|_{n/\alpha})t^{\frac{\alpha-1}{2}}T^{\frac{1}{2}(1-\frac{n}{m})}\|\phi\|_{\frac{n}{n-\alpha}}.
\end{aligned} \tag{5.5}$$

As in the proof of Theorem 2.1, by (4.4) and (5.5) we have that

$$K^T_{\alpha,j+1} \le K^T_{\alpha,0} + C_1(\alpha,n)(K^T_{\alpha,j})^2 + C_5\|w\|_{m,\infty;T}T^{\frac{1}{2}(1-\frac{n}{m})}K^T_{\alpha,j}. \tag{5.6}$$

Concerning (5.4), we have

$$\|\nabla u_0(t)\|_n \le M'_{n,n}\|a\|_n t^{-1/2}$$

$$\left\|\nabla \int_0^t e^{-(t-s)A}P(u_j\cdot\nabla u_j)(s)ds\right\|_n \le Q_{\frac{n}{\alpha+1}}M'_{\frac{n}{\alpha+1},n}K^T_{\alpha,j}L^T_j B(\tfrac{1-\alpha}{2},\tfrac{\alpha}{2})t^{-1/2}$$

$$\left\|\nabla \int_0^t e^{-(t-s)A}P(w\cdot\nabla u_j)(s)ds\right\|_n \le Q_{\frac{nm}{n+m}}M'_{\frac{nm}{n+m},n}\|w\|_{m,\infty}L^T_j B(\tfrac{1}{2}(1-\tfrac{n}{m}),\tfrac{1}{2})t^{-\frac{n}{2m}}$$

$$\left\|\nabla \int_0^t e^{-(t-s)A}P(u_j\cdot\nabla w)(s)ds\right\|_n \le Q_{\frac{n}{\alpha+1}}M'_{\frac{n}{\alpha+1},n}K^T_{\alpha,j}\|(\cdot)^{1/2}\nabla w\|_{n,\infty;T}B(\tfrac{1-\alpha}{2},\tfrac{\alpha}{2})t^{-1/2}$$

Hence (5.4) is true with j replaced by $j+1$, with

$$L^T_{j+1} \equiv M'_{n,n}\|a\|_n + Q_{\frac{n}{\alpha+1}}M'_{\frac{n}{\alpha+1},n}B(\tfrac{1-\alpha}{2},\tfrac{\alpha}{2})K^T_{\alpha,j}\|(\cdot)^{1/2}\nabla w\|_{n,\infty;T} + C_4(K^T_{\alpha,j}+\|w\|_{m,\infty}T^{\frac{1}{2}(1-\frac{n}{m})})L^T_j, \tag{5.7}$$

where $C_4 = Q_{\frac{n}{\alpha+1}}M'_{\frac{n}{\alpha+1},n}B(\frac{1-\alpha}{2},\frac{\alpha}{2}) + Q_{\frac{nm}{n+m}}M'_{\frac{nm}{n+m},n}B(\frac{1}{2}(1-\frac{n}{m}),\frac{1}{2})$. Therefore by induction, we get (5.3) and (5.4) for $j=0,1,\ldots$. Let $C_6(T) = 1 - C_5\|w\|_{m,\infty}T^{\frac{1}{2}(1-\frac{n}{m})}$. Since we may take $K^T_{\alpha,0}\equiv M_{\frac{n}{\alpha},\frac{n}{\alpha}}\|a\|_{\frac{n}{\alpha}}T^{\frac{1-\alpha}{2}}$, by (5.6) we have

$$K^T_{\alpha,j} < \frac{C_6(T) - \sqrt{(C_6(T))^2 - 4C_1M_{\frac{n}{\alpha},\frac{n}{\alpha}}\|a\|_{\frac{n}{\alpha}}T^{\frac{1-\alpha}{2}}}}{2C_1} \equiv k^T_\alpha, \quad j=0,1,\ldots, \tag{5.8}$$

provided

$$C_6(T) = 1 - C_5\|w\|_{m,\infty}T^{\frac{1}{2}(1-\frac{n}{m})} > 0, \tag{5.9}$$

$$4C_1M_{\frac{n}{\alpha},\frac{n}{\alpha}}\|a\|_{\frac{n}{\alpha}}T^{\frac{1-\alpha}{2}} < (1 - C_5\|w\|_{m,\infty}T^{\frac{1}{2}(1-\frac{n}{m})})^2. \tag{5.10}$$

Since $C_6(T^*) = 1 - C_5\|w\|_{m,\infty}T^{*\frac{1}{2}(1-\frac{n}{m})} > 1/2$, $4C_1M_{\frac{n}{\alpha},\frac{n}{\alpha}}\|a\|_{\frac{n}{\alpha}}T^{*\frac{1-\alpha}{2}} < 1/4$, T^* satisfies (5.9) and (5.10). Hence, as in the proof of Theorem 2.1, we obviously see that there is a limit $u\in C((0,T^*);L^{n/\alpha}_\sigma)$ with $t^{\frac{1-\alpha}{2}}u(\cdot)\in BC([0,T^*);L^{n/\alpha}_\sigma)$ stisfying

$$\sup_{0<t<T^*} t^{\frac{1-\alpha}{2}}\|u_j(t)-u(t)\|_{n/\alpha}\to 0 \text{ as } j\to\infty,$$

$$\sup_{0<t<T} t^{\frac{1-\alpha}{2}}\|u(t)\|_{n/\alpha}\to 0 \text{ as } T\to +0.$$

Moreover we shall show $t^{1/2}\nabla u(\cdot)\in L^\infty(0,T^*;L^n)$. (5.7) and (5.8) yield

$$L^{T^*}_{j+1} \le M'_{n,n}\|a\|_n + Q_{\frac{n}{\alpha+1}}M'_{\frac{n}{\alpha+1},n}B(\tfrac{1-\alpha}{2},\tfrac{\alpha}{2})k^{T^*}_\alpha\|(\cdot)^{1/2}\nabla w\|_{n,\infty;T^*} + C_4(k^{T^*}_\alpha + \|w\|_{m,\infty}T^{*\frac{1}{2}(1-\frac{n}{m})})L^{T^*}_j. \tag{5.11}$$

We can see $C_4k^{T^*}_\alpha < 1/2$ and $C_4\|w\|_{m,\infty}T^{*\frac{1}{2}(1-\frac{n}{m})} < 1/2$. Indeed if $\frac{1}{2C_4} > \frac{C_6(T^*)}{2C_1}$, then it follows from (5.8) that $k^{T^*}_\alpha < \frac{1}{2C_4}$. If $\frac{1}{2C_4} \le \frac{C_6(T^*)}{2C_1}$, *i.e.*, $C_6(T^*) - \frac{C_1}{C_4} \ge 0$, then it follows from the definition of T^* that

$$4C_1M_{\frac{n}{\alpha},\frac{n}{\alpha}}\|a\|_{n/\alpha}(T^*)^{\frac{1-\alpha}{2}} \le \frac{C_1}{4(C_1+C_4)} < \frac{C_1}{2C_4} < \frac{C_1}{C_4}C_6(T^*) \le \frac{C_1}{C_4}\left\{2C_6(T^*) - \frac{C_1}{C_4}\right\},$$

which yields

$$k_\alpha^{T^*} < \frac{C_6(T^*) - \sqrt{(C_6(T^*) - \frac{C_1}{C_4})^2}}{2C_1} = \frac{1}{2C_4}.$$

By the definition of T^* we obviously have $C_4\|w\|_{m,\infty}T^{*\frac{1}{2}(1-\frac{n}{m})} < 1/2$. Thus we obtain

$$C_4(k_\alpha^{T^*} + \|w\|_{m,\infty}T^{*\frac{1}{2}(1-\frac{n}{m})}) < 1. \tag{5.12}$$

Hence from (5.11) we see that the sequence $\{L_j^{T^*}\}_{j=0}^\infty$ is bounded with

$$L_j^{T^*} < \frac{M'_{n,n}\|a\|_n + Q_{\frac{n}{\alpha+1}} M'_{\frac{n}{\alpha+1},n} B(\frac{1-\alpha}{2}, \frac{\alpha}{2})\|(\cdot)^{1/2}\nabla w\|_{n,\infty;T^*}k_\alpha^{T^*}}{1 - C_4(k_\alpha^{T^*} + \|w\|_{m,\infty}T^{*\frac{1}{2}(1-\frac{n}{m})})} \equiv L^{T^*}. \tag{5.13}$$

By standerd argument, such a bound yields

$$t^{1/2}\nabla u(\cdot) \in L^\infty(0,T^*;L^n).$$

By (5.1) and (5.2) we easily show that $u_j \in C([0,T^*);L^n_\sigma)$ for $j = 0,1,\ldots$. In the similar way to proving (5.11), we have

$$\begin{aligned}\sup_{0<t<T^*}\|u_{j+1}\|_n \;\le\; & M_{n,n}\|a\|_n + Q_{\frac{n}{\alpha+1}}M_{\frac{n}{\alpha+1},n}B(1-\tfrac{\alpha}{2},\tfrac{\alpha}{2})k_\alpha^{T^*}(L^{T^*} + \|(\cdot)^{\frac{1}{2}}\nabla w\|_{n,\infty;T^*}) \\ & + Q_{\frac{mn}{m+n}}M_{\frac{mn}{m+n},n}B(1-\tfrac{n}{2m},\tfrac{1}{2})\|w\|_{m,\infty;T^*}L^{T^*}T^{*\frac{1}{2}(1-\frac{n}{m})}\end{aligned}$$

for $j = 0,1,\ldots$, which yields $u \in BC([0,T^*);L^n_\sigma)$. Hence as in the proof of Theorem 2.1, we see that u is a unique mild solution of (N-S') in the class $S_{n/\alpha}(0,T^*)$. It follows from (iii) of Definition 2 and integration by parts that

$$\begin{aligned}(u(t),\phi) \;=\; & (e^{-tA}a,\phi) - \int_0^t (e^{-(t-s)A}P(u\cdot\nabla u)(s),\phi)ds \\ & - \int_0^t (e^{-(t-s)A}P(w\cdot\nabla u)(s),\phi)ds - \int_0^t (e^{-(t-s)A}P(u\cdot\nabla w)(s),\phi)ds,\end{aligned}$$

for all $\phi \in C^\infty_{0,\sigma}$, all $0 < t < T^*$. It is easy show that $\int_0^t e^{-(t-s)A}P(u\cdot\nabla u)(s)ds$, $\int_0^t e^{-(t-s)A}P(w\cdot\nabla u)(s)ds$ and $\int_0^t e^{-(t-s)A}P(u\cdot\nabla w)(s)ds$ belong to L^n_σ for all $0 < t < T^*$. Thus we obtain

$$\begin{aligned}u(t) = & e^{-tA} - \int_0^t e^{-(t-s)A}P(u\cdot\nabla u)(s)ds \\ & - \int_0^t e^{-(t-s)A}P(w\cdot\nabla u)(s)ds - \int_0^t e^{-(t-s)A}P(u\cdot\nabla w)(s)ds \text{ in } L^n_\sigma,\end{aligned} \tag{5.14}$$

for $0 < t < T^*$. Next we shall show that this mild solution u is actually a strong solution if w satisfies, for some $\kappa \in (0,1)$, $w \in C^\kappa([\xi,T^*];L^\infty)$, $\nabla w \in C^\kappa([\xi,T^*];L^n)$ for all $\xi \in (0,T^*)$.

Since $w \in L^\infty(0,T^*;L^m)$ implies that $\sup_{0<s<T^*} s^{\frac{1-\delta}{2}}\|w(s)\|_{n/\delta} < \infty$ for $\delta = n/m$, by (5.14) we have $\sup_{0<s<T^*} s^{\frac{1-\delta}{2}}\|u(s)\|_{n/\delta} < \infty$. As in [12, Lemma A.4], from Lemmas 3.1 and 3.2 we obtain for $\kappa' > 0$ with $0 < \delta/2 + \kappa' < 1/2$,

$$\|u(t+h) - u(t)\|_\infty \;\le\; M(h^{\kappa'}t^{-\frac{1}{2}-\kappa'} + h^{\frac{1}{2}-\frac{\delta}{2}}t^{-1+\frac{\delta}{2}}), \tag{5.15}$$

$$\|\nabla u(t+h) - \nabla u(t)\|_n \;\le\; M(h^{\kappa'}t^{-\frac{1}{2}-\kappa'} + h^{\frac{1}{2}-\frac{\delta}{2}}t^{-1+\frac{\delta}{2}}), \tag{5.16}$$

for all $0 < t < t+h < T^*$. From these estimates and the hypotheses on w it follows that, for some $\kappa_0 > 0$,

$$u \cdot \nabla u, \quad w \cdot \nabla u, \quad u \cdot \nabla w \in C^{\kappa_0}([\xi, T^*]; L^n)$$

for all $\xi \in (0, T^*)$. Then a well-known theory of holomorphic semigroup states that u is a strong solution of $(N-S')$ on $(0, T^*)$ (see, e.g., Tanabe [16, Theorem 3.3.4]). This completes the proof of Theorem 5.1.

Proof of Theorem 2.3. Let w is a strong solution of $(N-S)$ for some $f \in C(0,\infty; L^n_\sigma)$. Since w is a strong solution of $(N-S)$ on $(0,\infty)$, we have $\nabla w \in L^\infty(\epsilon, T; L^n)$ for all $0 < \epsilon < T < \infty$, which implies

$$t^{1/2}\nabla w(\cdot + \epsilon) \in L^\infty_{loc}([0,\infty); L^n).$$

Moreover, as in [12, Lemma A.4], from Lemmas 3.1 and 3.2 we obtain for some $\kappa \in (0,1)$,

$$w \in C^\kappa([\xi, T]; L^\infty), \quad \nabla w \in C^\kappa([\xi, T]; L^n) \tag{5.17}$$

for all $0 < \epsilon < \xi < T < \infty$. Since u is the mild solution in the class $S_{2n}(0,\infty)$, we have

$$\sup_{s \geq \epsilon} \|u(s)\|_{2n} \leq A_\epsilon < \infty \text{ for } \epsilon > 0.$$

Letting $\alpha = 1/2$ and

$$T^*_\epsilon = \min\left\{ \left[\frac{1}{16(C_1 + C_4)M_{2n,2n}A_\epsilon}\right]^4, \left(\frac{1}{2(C_4 + C_5)\|w\|_{m_2,\infty}}\right)^{\frac{2m_2}{m_2-n}} \right\},$$

by Lemma 3.5, Lemma 3.6 and Theorem 5.1 we see that u is a strong solution on all interval $(t, t+T^*_\epsilon) \subset (\epsilon, \infty)$. Hence we conclude by standard argument that u is a strong solution on (ϵ, ∞). Since $\epsilon > 0$ is arbitrary, this completes the proof of Theorem 2.3.

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